B. Sc. (Hons) Part I

1st Paper Trigonometry

Hyperbolic functions contd.

Q. If $\cos(\theta + i\phi)\cos(\alpha + i\beta) = 1$, prove that $\tanh^2\phi \cosh^2\beta = \sin^2\alpha$ and $\tanh^2\beta \cosh^2\phi = \sin^2\theta$.

Soln. Given that

$$\cos(\theta + i\phi)\cos(\alpha + i\beta) = 1$$

$$\Rightarrow \cos(\theta + i\phi) = \sec(\alpha + i\beta) \quad \text{--- (1)}$$

Putting $-i$ in place of i in (1), we have

$$\cos(\theta - i\phi) = \sec(\alpha - i\beta) \quad \text{--- (2)}.$$

Now, $\sin(\theta + i\phi) = \sqrt{1 - \cos^2(\theta + i\phi)}$

$$\Rightarrow \sin(\theta + i\phi) = \sqrt{1 - \sec^2(\alpha + i\beta)} \quad [\text{using (1)}]$$

$$= \sqrt{-\tan^2(\alpha + i\beta)}$$

$$\Rightarrow \sin(\theta + i\phi) = i\tan(\alpha + i\beta) \quad \text{--- (3)}$$

Replacing i by $-i$, we have

$$\sin(\theta - i\phi) = -i\tan(\alpha - i\beta) \quad \text{--- (4)}$$

Now, $2i\phi = (\theta + i\phi) - (\theta - i\phi)$

$$\Rightarrow \cos 2i\phi = \cos[(\theta + i\phi) - (\theta - i\phi)]$$

$$\Rightarrow \cos 2i\phi = \cos(\theta+i\phi)\cos(\theta-i\phi) + \sin(\theta+i\phi)\sin(\theta-i\phi)$$

$$\Rightarrow \cos 2i\phi = \sec(\alpha+i\beta)\sec(\alpha-i\beta) + (i\tan(\alpha+i\beta)).(-i\tan(\alpha-i\beta)).$$

$$\Rightarrow \cos 2i\phi = \frac{1}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)} + \frac{\sin(\alpha+i\beta)\sin(\alpha-i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$$

$$\Rightarrow \frac{\cosh 2\phi}{1} = \frac{1+\sin(\alpha+i\beta)\sin(\alpha-i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)} = \frac{1+\sin^2\alpha-\sin^2 i\beta}{\cos^2\alpha-\sin^2 i\beta}$$

By componendo & Dividendo

$$\Rightarrow \frac{\cosh 2\phi - 1}{\cosh 2\phi + 1} = \frac{1+\sin^2\alpha - \cancel{\sin^2 i\beta} - \cos^2\alpha + \cancel{\sin^2 i\beta}}{1+\sin^2\alpha - \sin^2 i\beta + \cos^2\alpha - \sin^2 i\beta}$$

$$\Rightarrow \frac{2\sinh^2\phi}{2\cosh^2\phi} = \frac{(1-\cos^2\alpha)+\sin^2\alpha}{1+(\sin^2\alpha+\cos^2\alpha)-2\sin^2 i\beta}$$

$$\Rightarrow \tanh^2\phi = \frac{2\sin^2\alpha}{2(1-\sin^2 i\beta)} = \frac{\sin^2\alpha}{\cosh^2 i\beta}$$

$$\Rightarrow \tanh^2\phi = \frac{\sin^2\alpha}{\cosh^2\beta}$$

$$\Rightarrow \tanh^2\phi \cosh^2\beta = \sin^2\alpha \quad \text{Proved}$$

Replacing θ by α, ϕ by β, α by θ, β by ϕ, in the above, we get

$$\tanh^2\beta \cosh^2\phi = \sin^2\theta \quad \text{Proved}$$